Extensions of the Zwart-Powell Box Spline for Volumetric Data Reconstruction on the Cartesian Lattice

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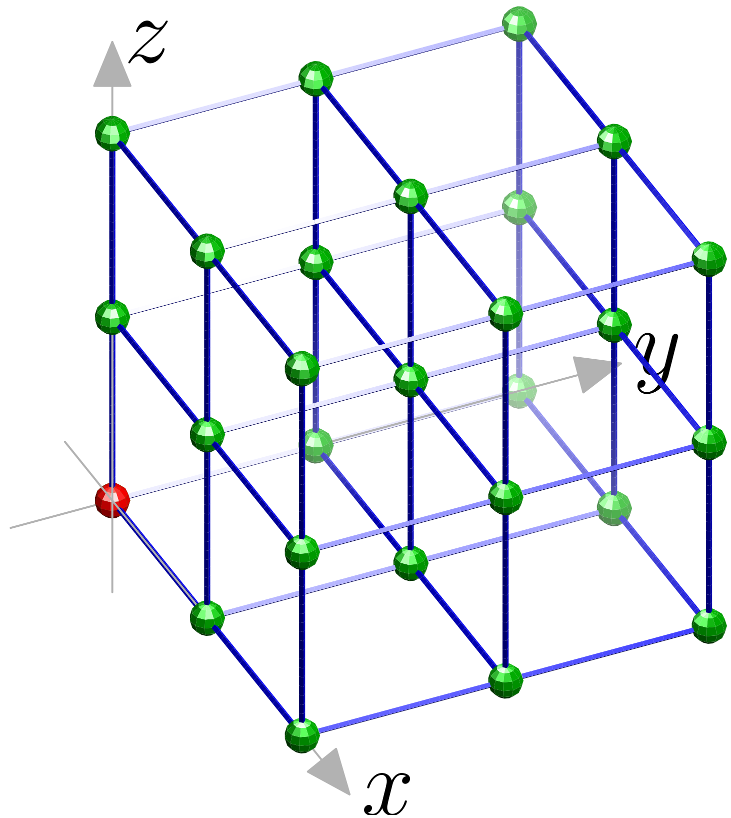
Reconstruction on the Cartesian Lattice

A ubiquitous problem in many computing areas:

- Scientific computing
- Visualization
- Computer Graphics
- ...



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essentially 1-D solutions, not true multi-D solutions. Our approach: true 3-D

Our Reconstruction Algorithm

tri-linear B-spline

tri-cubic B-spline

box spline Ξ

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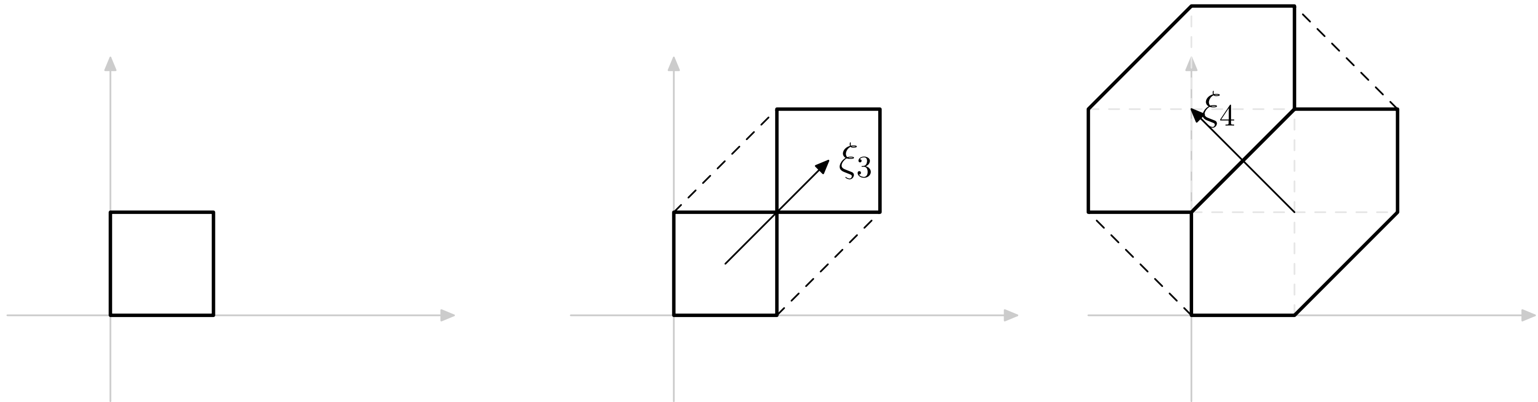

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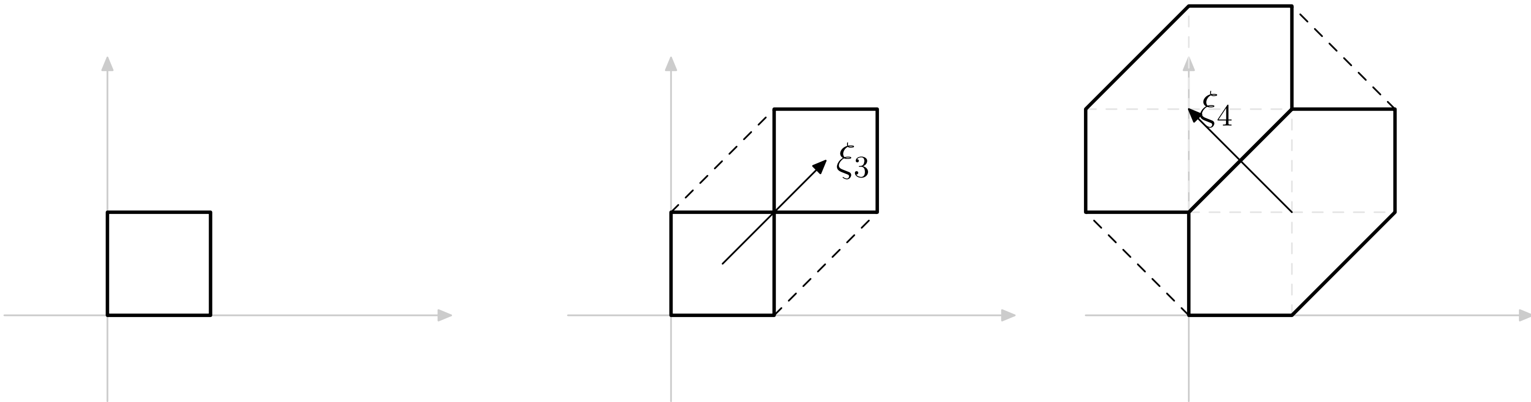

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- *Zwart-Powell element*

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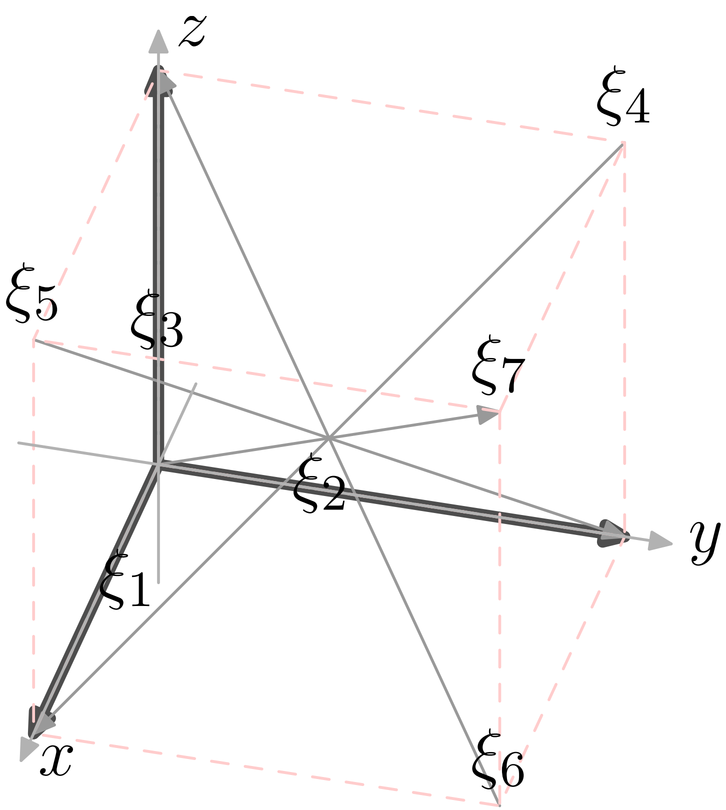
- Extend the Zwart-Powell element to tri-variate setting
- 7-directional box spline

Seven Directional Box Spline

- 3-axis aligned directions
- 4-diagonal directions



Seven Directional Box Spline

- Matrix of directions:

$$\boldsymbol{\Xi} = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 1 \end{bmatrix} \tag{1}$$

- $\rho = \min\{Z \in \boldsymbol{\Xi}, \boldsymbol{\Xi} \backslash Z$ does not span $\mathbb{R}^3\} \therefore \rho = 4$.
- $M_{\boldsymbol{\Xi}} \in C^{\rho-2} = C^2$

Approximation Power

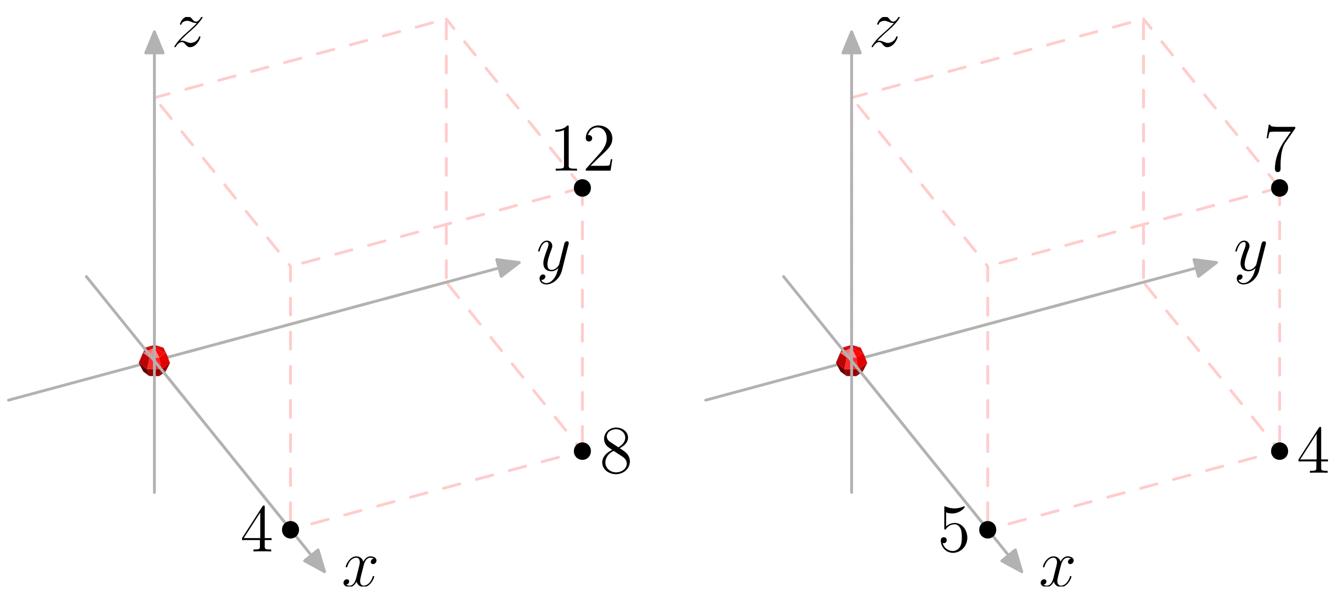
- Fourier Transform:

$$\hat{M}_{\boldsymbol{\Xi}}(\boldsymbol{\omega}) = \prod_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} \operatorname{sinc}(\boldsymbol{\xi} \cdot \boldsymbol{\omega}) \tag{2}$$

- Essentially product of *directional* sinc's
- One can verify zero-crossing of $\hat{M}_{\boldsymbol{\Xi}}$

Approximation Power

- Number of Vanishing moments



tri-cubic B-spline — box spline M_{Ξ}

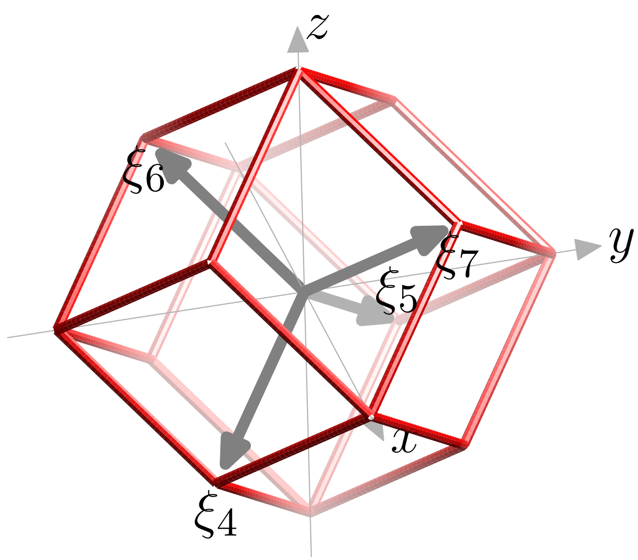
- A minimum of *four* vanishing moments.
- Approximation Power = 4

B-splines?

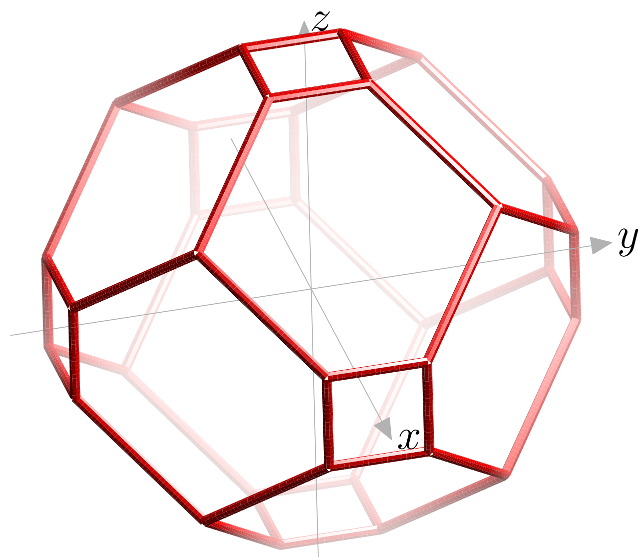
The smoothness and approximation power matches that of the *tri-cubic B-spline*.

Support

- Convolution of a 4-directional box spline with a cube



4-dir



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- The support of this box spline is *contained* in $5 \times 5 \times 5$ neighborhood.
- Originally we considered this box spline to be slower
- It turns out that only 53 points fall inside the support.
- 20% *faster than tri-cubic B-spline*

Numerical Implementation

- Table look up for box spline values
- The box spline $M_{\boldsymbol{\Xi}}$ can be decomposed into

$$M_{\boldsymbol{\Xi}}(\boldsymbol{x}) = \left(M_{\boldsymbol{\Xi}_1} * M_{\boldsymbol{\Xi}_2}\right)(\boldsymbol{x})$$

where

$$\boldsymbol{\Xi}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{\Xi}_2 = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

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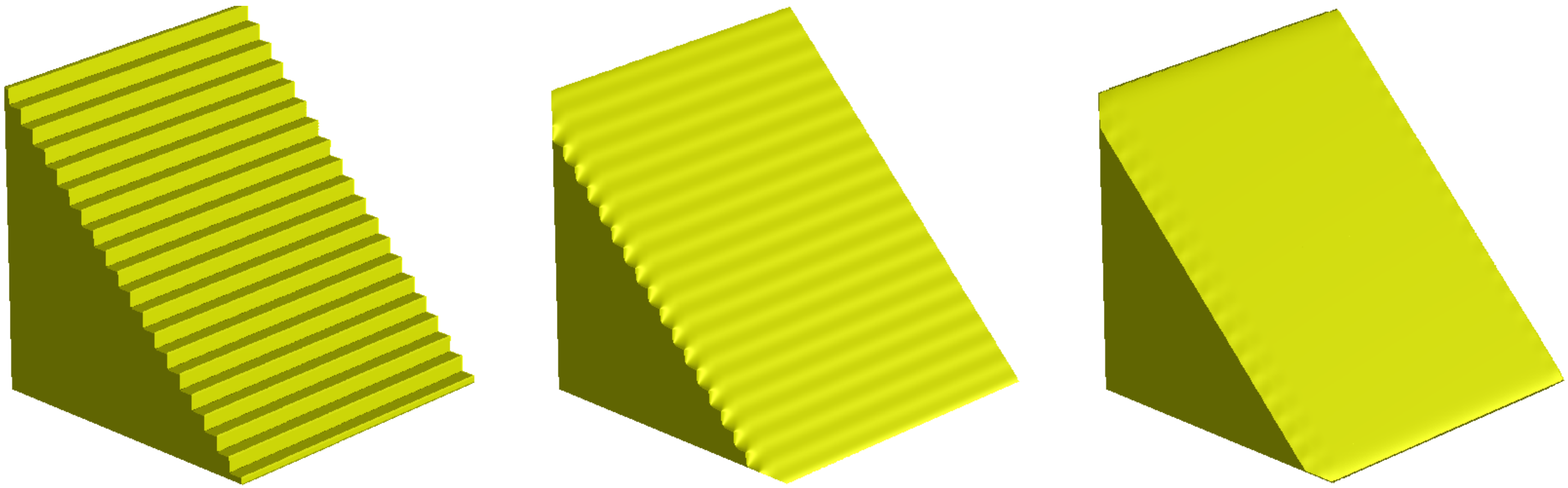
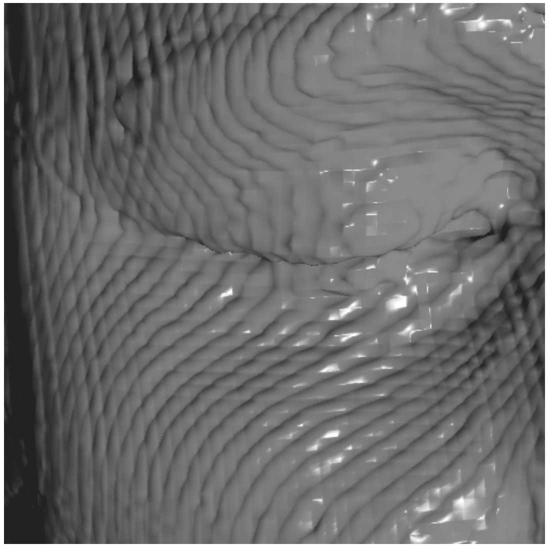
- Can be efficiently performed by multiplication in Fourier domain

Stair-casing in Recontruction



Voxelized surface | tri-cubic B-spline | box spline Ξ

Grid-Aligned Artifacts



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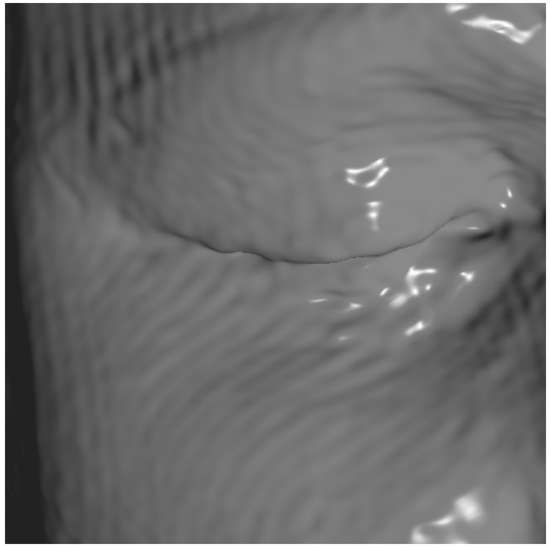

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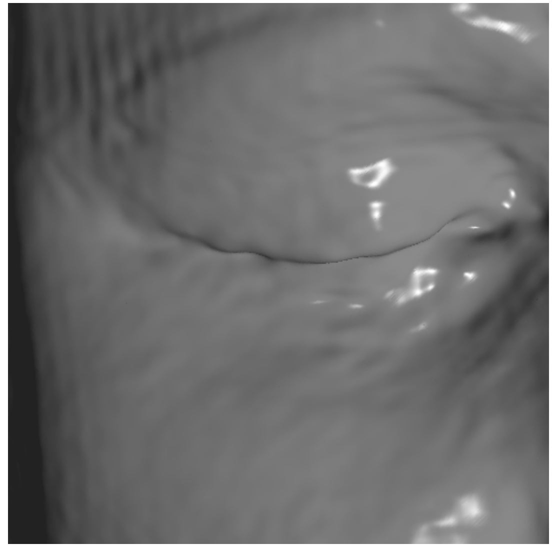

box spline M_{Ξ}

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Conclusion

The 7-directional box spline

- has the same smoothness and approximation power of the tri-cubic B-spline
- offers a more isotropic treatment of data and reduces axis-aligned artifacts
- is computationally more efficient than tri-cubic B-spline (20%)